1/N-Energy-Solutions to Relativistic Two-Boson Systems with an Exponential Potential

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Closed $1/N$-energy-formulae for relativistic two-boson ($m_1 \neq m_2$) systems with a vector exponential potential have been written down to first $1/N$-order. Numerical estimates and structural properties are presented.

Key words: Quantum mechanical $1/N$-method, Relativistic two-body spectroscopy, Exponential potential, Bound-state energies, Mass-asymmetry ($m_1 \neq m_2$) properties.

1. Introduction

Nonperturbative and quickly tractable approximations to the energies of two-body relativistic systems for which the classical Hamiltonian reads

$$\mathscr{H}(x, p) = (p^2 + m_1^2)^{1/2} + (p^2 + m_2^2)^{1/2} + U(x) = \mathscr{E}, \quad (1)$$

have been performed recently [1] with the help of the $1/N$-method [2]. Here $p = |\boldsymbol{p}|$ denotes the momentum-magnitude in the center-of-mass system, $x = |\boldsymbol{x}|$ is the relative separation and $\mathscr{E}$ represents the energy. It should be specified that all the above variables are dimensionless and $\hbar = c = 1$. As a matter of fact, the energies derived in this manner exhibit a good accuracy for two-boson systems [3]. Certain leading-extrapolations towards accounting for spin-effects have also been proposed for Coulomb-like systems [1, 3], but further refinements are in order. The above method has also been proven to be quite useful for the description of relativistic meson-spectra [3] in terms of the familiar linear plus Coulomb quark-antiquark potential [4], now for $m_1 \neq m_2$. In particular, the energies of relativistic single-particle systems with Coulomb plus Aharonov-Bohm potentials can be derived in a similar way [5]. In this paper we shall then continue such studies by discussing the energy-spectrum of the exponential potential

$$U(x) = -g \exp(-\lambda x), \quad (2)$$

where $g > 0$ and $\lambda > 0$, which is of interest in the description of the deuteron. The main interest concerns, however, typical structural properties and the relativistic two-body attributes of the exponential binding.

2. The Derivation of 1/N-Energy-Formulae

Squaring twice (1) leads to an effective Schrödinger-Hamiltonian like

$$\mathscr{H}_{\text{eff}}(x, p) = p^2 + V(x, \mathscr{E}) = E(\mathscr{E}), \quad (3)$$

in which the energy-dependent potential reads [1, 6]

$$V(x, \mathscr{E}) = \tfrac{1}{2}\mathscr{E}U - \tfrac{1}{4}U^2 - \frac{\delta}{4}(\mathscr{E} - U)^{-2}, \quad (4)$$

where

$$E(\mathscr{E}) = \tfrac{1}{4}(\mathscr{E}^2 - M), \quad (5)$$

and $M = 2(m_1^2 + m_2^2)$. The mass-asymmetry-parameter is $\delta = (m_1^2 - m_2^2)^2$, whereas $\mu = m_1 + m_2$ stands for the kinematical threshold. This opens the way to perform the quasiclassical minimization of $\mathscr{H}_{\text{eff}}(x, p)$, i.e. the minimization of $\mathscr{H}_{\text{eff}}(x, d_0/x)$, as discussed before [7]. Then the energy of (3) is given by

$$E(\mathscr{E}) = \min \mathscr{H}_{\text{eff}}\left(x, \frac{d_0}{x}\right) = V(x_0) + \frac{x_0}{2}V'(x_0), \quad (6)$$

in which d_0 expresses the expansion-parameter of the $1/N$-method, while $x = x_0$ denotes the location of the minimum. Next it is convenient to introduce the variables $z = \mu x_0$ and [1]

$$u = \mathscr{E} + g \exp(-z) > \mu, \quad (7)$$

so that

$$0 < f \equiv \frac{u^4 - Mu^2 + \delta}{u^4 - \delta} < 1. \quad (8)$$

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Under such conditions one obtains the coupled algebraic-equations

$$\frac{1}{z}\exp z=\frac{g}{u\,f}\,, \tag{9}$$

and

$$d_0^2=z^2\,\frac{u^4-M\,u^2+\delta}{4\,\lambda^2\,u^2}\,, \tag{10}$$

such that [7]

$$d_0=L+\mathcal{N}\Omega^{1/2}\,, \tag{11}$$

to first $1/N$-order, where

$$\Omega=3-z-f\,\frac{u^4+3\,\delta}{u^4-\delta}>0\,. \tag{12}$$

One has $\mathcal{N}=n_r+1/2$ and $L=l+(N-2)/2$, where N is the number of space-dimensions, $n_r=0, 1, 2, \ldots$ is the radial quantum-number and $l=0, 1, 2, \ldots$ is the quantum-number of the angular-momentum, as usual. We have to mention that Ω is subject to positive values by virtue of the pertinent concavity-condition. Now all that remains is to solve the coupled algebraic equations (9)–(11). Having obtained u and z then gives the energy as

$$\mathscr{E}=\mathscr{E}(g;\,l,\,n_r)=u\left(1-\frac{f}{z}\right), \tag{13}$$

in accord with (5)–(6). However, we have to realize that the easiest way to solve (9)–(11) is to consider the energy $\mathscr{E}$ as an input parameter. Indeed, starting from $\mathscr{E}$-inputs yields

$$z=\frac{u}{u-\mathscr{E}}\,f\,, \tag{14}$$

which in turn leads to the conversion of the initial energy eigenvalue-problem into the evaluation of the energy-dependent coupling:

$$g=g(\mathscr{E})=g(\mathscr{E};\,l,\,n_r)\,. \tag{15}$$

The point is that (15) proceeds just in terms of a single algebraic equation concerning the u-variable, which can be easily solved with the help of a small computer. In other words, we are led to a simplified conjugate eigenvalue-problem, as anticipated above.

3. Typical Properties and Numerical Results

We have to realize that z increases monotonically with $\mathscr{E}$, so that one has $z=1$ for a characteristic energy located to the left of the kinematical threshold $\mathscr{E}=\mu$. In this context, $\mathscr{E}$ becomes subject to arbitrary negative values going unlimitedly beyond $\mathscr{E}=-\mu$. Indeed, one has $z<1$ as soon as $\mathscr{E}<0$, so that $\mathscr{E}\to-\infty$ for $z\to 0$, as shown by (13). It should also be noted that the above $1/N$-equations lead specifically to positive-energy excitations, i.e. to energies which are larger than μ. However, such energies are confined within a quite small strip $\mathscr{E}\in(\mu,\,\mu+\varepsilon)$, such that $\mu\gg\varepsilon$. Other typical structural properties can also be established. So the concavity-limit $\Omega\to 0$ yields upper z-bounds like

$$z\to z_c=3-f\,\frac{u^4+3\,\delta}{u^4-\delta}\,, \tag{16}$$

which produce l-dependent lower-bounds, say $g_c(\lambda,\,l)$, on the g-coupling via $d_0\to L$. On the other hand, $g(\mathscr{E};\,l,\,n_r)$ increases with l and n_r, so that these lower-bounds are of an immediate concern for the ground-states ($n_r=0$). Further, $g(\mathscr{E})$ decreases monotonically as $\mathscr{E}$ increases. This means that the present conjugate eigenvalue-problem exhibits bound-state-solutions only for subcritical energies, which agrees with the lower g-bounds mentioned before. Such subcritical energies are, however, larger than μ, which explains the onset of positive-energy excitations referred to above.

Putting $N=3$ and choosing, for instance, $m_1=2$ and $m_2=1$ ($\delta=9$ and $M=10$), we have established numerical estimates for the critical couplings $g_c(\lambda,\,l)$ and for the corresponding energies $\mathscr{E}_c(\lambda,\,l)$, as shown in Table 1. We have restricted ourselves to $0.1\le\lambda\le 2$ and to $l=0, 1$. The λ-dependence displayed in Fig. 1 enables us to say that $g_c(\lambda,\,l)$ is a concave and monotonically increasing function of λ. The $g=g(\mathscr{E})$-solutions have been presented in Table 2, now for

Table 1. Estimates of the critical couplings $g_c(\lambda,\,l=0)$ and $g_c(\lambda,\,l=1)$ for several λ-values. Corresponding energies have also been inserted. The mass-values have been chosen as $m_1=2$ and $m_2=1$.

λ	$g_c(\lambda,\,0)$	$\mathscr{E}_c(\lambda,\,0)$	$g_c(\lambda,\,1)$	$\mathscr{E}_c(\lambda,\,1)$
0.1	0.002789	3.000069	0.025083	3.000625
0.2	0.011154	3.000279	0.100053	3.002505
0.4	0.044560	3.001112	0.395789	3.010074
0.6	0.100053	3.002505	0.874517	3.022866
0.8	0.177356	3.004459	1.516896	3.041124
1.0	0.276096	3.006980	2.299341	3.065137
1.2	0.395789	3.010074	3.196742	3.095192
1.6	0.695690	3.018008	5.242521	3.174265
2.0	1.071563	3.028331	7.497896	3.278962

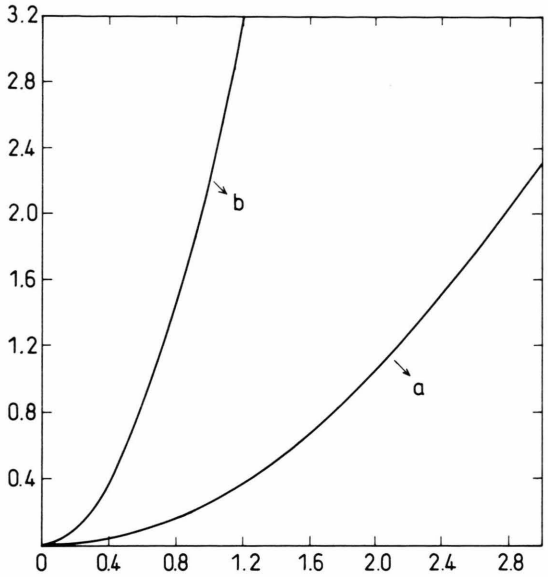

Fig. 1. The λ-dependence of the critical coupling $g_c(\lambda, l)$ for $l=0$ (a) and $l=1$ (b).

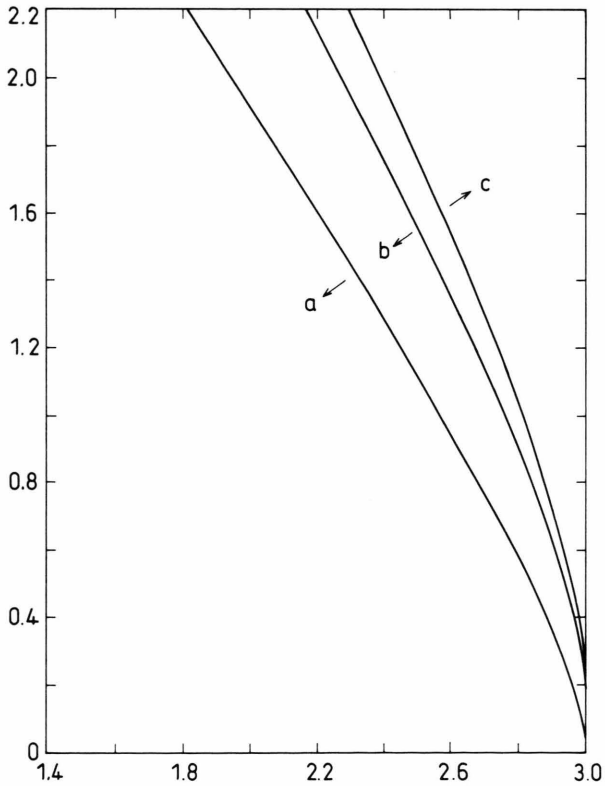

Fig. 2. The coupling $g(\mathscr{E}; l, n_r)$ versus the energy $\mathscr{E}$ for $l=n_r=0$ (a), $l=1$ and $n_r=0$ (b) as well as for $l=0$ and $n_r=1$ (c).

Table 2. Solutions to the exponential coupling $g(\mathscr{E}; l, n_r)$ for $-4 \leq \mathscr{E} \simeq 3$, $\lambda=0.2$, $m_1=2$ and $m_2=1$.

$\mathscr{E}$	$l=n_r=0$	$l=1, n_r=0$	$l=0, n_r=1$
3.00	0.055212	0.219237	0.217678
2.95	0.252927	0.476952	0.550341
2.90	0.377709	0.642736	0.743176
2.80	0.589479	0.914871	1.050423
2.60	0.957148	1.370371	1.551988
2.40	1.291980	1.773841	1.988956
2.00	1.914705	2.506951	2.772892
1.60	2.502460	3.185509	3.491170
1.00	3.346323	4.144913	4.499059
0.00	4.692698	5.651747	6.070269
−1.00	5.992823	7.088136	7.559219
−2.00	7.262441	8.478361	8.994669
−3.00	8.509862	9.835166	10.391595
−4.00	9.740135	11.166244	11.759014

Table 3. Solutions to the exponential coupling for positive energy excitations for which $\mu=3$, $n_r=0$ and $l=0, 1$.

$\mathscr{E}$	$g(\mathscr{E}; 0, 0)$	$\mathscr{E}$	$g(\mathscr{E}; 1, 0)$
3.00010	0.053689	3.0005	0.213626
3.00020	0.052033	3.0010	0.207559
3.00030	0.050202	3.0020	0.193304
3.00040	0.048120	3.0030	0.172127
3.00050	0.045639	3.0031	0.168846
3.00060	0.042374	3.0032	0.164912
3.00069	0.036774	3.0033	0.159643

$-4 \leq \mathscr{E} \leq 3$ and $\lambda=0.2$. The first three states, i.e. $l=n_r=0$, $l=1$ and $n_r=0$, as well as $l=0$ and $n_r=1$, have been chosen. Now we have to say that $g=g(\mathscr{E})$ is a convex function which decreases monotonically with $\mathscr{E}$, as shown in Figure 2. Positive-energy excitations have been presented in Table 3. We have to realize that the couplings and energies presented in this Table become subject progressively to the critical limits presented for $\lambda=0.2$ in Table 2. In addition, we remark that the positive energy-strips are quite narrow, as already noted before.

So far the above results lead to the ordering-rule

$$\mathscr{E}(g; l_1, n_r^{(1)}) < \mathscr{E}(g; l_2, n_r^{(2)}), \tag{17}$$

if $l_1 > l_2$, where $l_1 + n_r^{(1)} = l_2 + n_r^{(2)}$. However, $\Delta V(x)$ fails to exhibit, this time, a definite sign irrespective of $0 < x < \infty$ [8], which means that (17) cannot be extended directly to the related E-levels. Confrontations with available single-particle data [9] can also be done. Setting $m_1=m_2=1$ ($\delta=0$ and $M=4$) and $g=2Z\alpha$, one then finds that $\mathscr{E}=-2$ for $g \cong 11.438773$ ($g \cong 34.637274$) if $\lambda=1$ ($\lambda=5$). This reproduces satis-

factorily (reasonably) the numerical results $g \cong 11.354682$ ($g \cong 31.495378$) established before [9].

4. Conclusions

Proofs have been given that the energy-spectrum of the relativistic two-body system with a vector exponential potential can be established in a closed form, such as given, to first $1/N$-order, by the coupled algebraic equations (9)–(11). Energy-formulae to second $1/N$-order can also be easily established in a similar manner, using e.g. (4) from [7]. Looking for computational simplification, we have solved the present $1/N$-equations by applying the conjugate eigenvalue-method. We then found that such a method is appropriate for the description of structural properties, such as critical points of spectral curves, or positive-energy excitations.

Comparison with the energies of the Yukawa-potential

$$U(x) = -\frac{g}{x}\exp(-\lambda x)\,, \tag{18}$$

presented before [1], can also be made. First, the Yukawa-energies are subject to positive lower-bounds, in contradistinction to the exponential ones, which are unbounded from below. Second, both the critical couplings $g_c(\lambda, l)$ as well as the $g = g(\mathscr{E})$-solutions characterizing (18) exhibit critical terminating points indicating the onset of resonances. Such terminating points are not produced by the exponential potential, as shown in Figs. 1 and 2. However, the positive-energy excitations exhibit a similar behaviour in both cases.

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